Discrete convexity and packages

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Theory of convexity for the lattice of integer points $\mathbb{Z}^n$ allows us to answer to the questions

1) What subsets $X \subset \mathbb{Z}^n$ could be called "convex"?

2) What functions $F : \mathbb{Z}^n \to \mathbb{R}(\mathbb{Z})$ could be called "convex"?

One property of convexity of sets seems indisputable: X should coincide with the set of all integer points of its convex hull $\mathrm{co}(X)$. We call such sets pseudo-convex. The resulting class $\mathcal{P}C$ of all pseudo-convex sets is stable under intersection but not under summation. In other words, the sum $X + Y = \{x + y \mid x \in X, y \in Y\}$ of pseudo-convex sets X and Y needs not be pseudo-convex.

Example. Consider pseudo-convex sets $A = \{(0,0), (1,1)\}$ and $B = \{(0,0), (-1,1)\}$. Then $A + B = \{(0,0), (1,1), (-1,1), (0,2)\}$, while $co(A + B)$ contains one more integer point $(0,1)$. $\square$

We should consider subclasses of $\mathcal{PC}$ in order to obtain stability under summation. Stability under summation is closely related to another question: when the intersection of two integer polytopes is an integer polytope? We say that a class $\mathcal{K} \subset \mathcal{PC}$ is <u>ample</u> if $\mathcal{K}$ is stable under a) integer translations, b) reflection, and c) taking faces. In the same way we understand ampleness of a class of integer polytopes.

Theorem

Let $\mathcal{K} \subset \mathcal{PC}$ be an ample class. The following four properties of $\mathcal{K}$ are equivalent:
(Add) for every $X, Y \in \mathcal{K}$ the sets $X \pm Y$ are pseudo-convex;
(Sep) if sets X and Y of $\mathcal{K}$ do not intersect, then there exists (integer) linear functional $p : V \longrightarrow \mathbb{R}$ such that $p(x) > p(y)$ for any $x \in X$, $y \in Y$;
(Int) if sets X and Y of $\mathcal{K}$ do not intersect, then the polyhedra $co(X)$ and $co(Y)$ do not intersect as well;
(Edm) for every $X, Y \in \mathcal{K}$ the polyhedron $co(X) \cap co(Y)$ is integer.

Note that for three sets or polytopes the statement is not true in general.

When the intersection of two integer polytopes is an integer polytope? There are two important basic results. The first one is The matroids intersection theorem by Jack Edmonds (1967).

The intersection of two matroid polytopes is an integer polytope (not need to be a matroid polytope).

Recall that a matroid is a combinatorial abstraction of the linear independence. Specifically, a collection of bases $\mathcal{M} \subset 2^{[n]}$, $[n] := \{1, \ldots, n\}$, is a matroid if, for any A, $B \in \mathcal{M}$, and $a \in A \setminus B$ there exists $b \in B \setminus A$, such that $(A \cup b \setminus a)$ belongs to $\mathcal{M}$. The matroid polytope is the convex hull of the characteristic sets of the bases of $\mathcal{M}$,$co(\mathcal{M}) \subset [0, 1]^{[n]}$.
The theorem says that, for matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, the intersection the convex hulls of the corresponding bases is an integer polytope.

Tomizawa (1980) (in Japanese) and Gelfand and Serganova (1987) (in Russian)

independently discovered that the exchange axiom for matroids on the ground set $[n] := \{1, \ldots, n\}$ is nothing else but the statement that every matroid polytope is an integer polytope of the unit cube $[0, 1]^{[n]}$ whose **edges parallel to the vectors of the set**

$$A_n := \{e_i - e_j \,|, i, j = 1, \ldots, n\}.$$

A_n is an important example of totally unimodular set of vectors, and at the same time is the set of positive roots for $\mathfrak{gl}_n$.

Recall that a collection U of vectors in $\mathbb{R}^n$ is totally unimodular if any subcollection $U' \subset U$ of linear independent vectors is a basis of the integer lattice $\mathbb{Z}^n \cap \mathbb{R}U'$, where $\mathbb{R}U'$ denotes the linear space generated by vectors in U', $\mathbb{R}U' = \{\sum_{u \in U'} \alpha_u u, \,|\, \alpha_u \in \mathbb{R}\}$.

Here is a simple proof of the matroid intersection theorem, which relies on total unimodularity of A_n.
Let x be a vertex of $M_1 \cap M_2$, where M_i, $i = 1, 2$, are matroid polytopes. Let F_1 and F_2 be faces of M_1 and M_2 of complementary dimensions such that $x \in F_1 \cap F_2$. Wlog we assume M_1 and M_2 are of complementary dimensions. Then F_i belongs to $f_i + \mathbb{R}U_i$, $i = 1, 2$, where f_i is a vertex of M_i and U_i is a subcollection of A_n of linear independent directions of edges of F_i. Then $U_1 \cup U_2$ is linear independent sub-collection of A_n. Therefore, due to the totally unimodularity of A_n, $f_1 - f_2$ is an integer linear combination of vectors of $U_1 \cup U_2$. Hence $x - f_2$ is the part of this combination which involves vectors from U_2. This implies that x is integer. $\square$

For a totally unimodular system U, let $Edm(U)$ be the class of integer polytopes, such that edges of each polytope of $Edm(U)$ are parallel to vectors of U.
Then the same arguments as in the above proof give us the following

Theorem

Let $P_1, P_2 \in Edm(U)$. Then $P_1 \cap P_2$ is an integer polytope.

Note that intersection of three polytopes P_1, P_2 and $P_3 \in Edm(U)$ might be not integer, in general. Due to the definition, $Edm(U)$ is stable under summations.

The second result on integrality of intersection of polytopes is due to Alan Hoffman and Joseph Kruskal (1956). They pointed out the importance of totally unimodular systems of vectors in optimization. Namely, they considered totally unimodular matrices, matrices with all minors in $\{-1, 0, 1\}$. Collection of rows (or columns) of such a matrix form a totally unimodular system of vectors in the space of corresponding dimension. Hoffman and Kruskal showed that LP problems of the form

$$\min_{x \geq 0, x^T A \leq b} c^T x$$

with integer vector b and a totally unimodular matrix (I, A) have integer solutions (one can get that from the Cramer rule).
Due to the ellipsoid method (due to Leonid Khachian), solutions to this problem can be found in polynomial time.

For a full-dimensional totally unimodular system U, let us consider a hyperplane arrangement

$$\mathcal{H}(U) = \{u^T x = b,\ u \in U,\ b \in \mathbb{Z}\}.$$

For example, for $U = A_n$, we get the χ-hyperplane arrangement in $\mathbb{R}^{n-1}$. In fact, let us choose $\hat{e}_{j-1} := e_1 - e_j$, $j = 2, \ldots, n$, as a basis, then

$$\{x_i - x_j = b,\ b \in \mathbb{Z},\ i < j\ x_i = a,\ a \in \mathbb{Z}\}$$

A <u>U-chamber</u> is a connected component of the complement to $\mathcal{H}(U)$ in $\mathbb{R}^n$, that is a connected component of $\mathbb{R}^n \setminus \mathcal{H}(U)$.
A <u>U-cell</u> is closure of a U-chamber. Faces of an U-cell we also call U-cells. The U-cells are integer polytopes, and they form a polyhedral complex covering $\mathbb{R}^n$.

For a totally unimodular system U, define a class $Hof(U)$ of integer polytopes constituted of integer polytopes such that normal vectors to facets belong to U. It is easy to see that a polytope of $Hof(U)$ is a union of U-cells.
Because U-cells form a polyhedral complex constituted of integer polytopes, we immediate get that for any $Q_1, Q_2 \in Hof(U)$

$$Q_1 \cap Q_2 \in Hof(U).$$

Because of this, the Edmonds intersection theorem holds true for the class $Hof(U)$. However, the Minkowski sum $Q_1 + Q_2$ of two polytopes of $Hof(U)$ might be outside of the class $Hof(U)$.

Let us note, that well-known class of the transportation polytopes is of the form $Hof(U)$. In fact, consider $n \times m$ transportation problem

$$\max_{x \in \mathbb{R}_+^{n \times m}} \; \sum_j x_{ij} \leq a_j, \sum_i x_{ij} \leq b_i \; c^T x.$$

The domains of such problems form the set $Hof(T_{n,m})$, where

$$T_{n,m} = \{e_{ij}, i \in [n], j \in [m], \sum_i e_{ij}, \; i \in [n], \sum_j e_{ij}, \; j \in [m]\}.$$

$T_{n,m}$ is a totally unimodular system (this follows, for example, from the Edmonds theorem on unimodularity of the union of two laminar collections).

Thus, for a totally unimodular system U, we have two classes $Edm(U)$ and $Hof(U)$ possessing the Edmonds intersection theorem. These classes look like dual:

vectors of U form directions for the edges of polyhedra of $Edm(U)$, while the vectors of U are normal vectors to facets of polyhedra of $Hof(U)$;

$P_1 + P_2 \in Edm(U)$, **but** $P_1 \cap P_2$ **can be not of** $Edm(U)$**;**

$Q_1 \cap Q_2 \in Hof(U)$, **but** $Q_1 + Q_2$ **can be not of** $Hof(U)$**.**

We establish the corresponding duality by the Legendre- Fenchel duality for discrete convex/concave functions.

Any polytope of the class $Hof(U)$ is a composition of U-cells. Since $Hof(U)$ is stable under intersections, for any pair of integer points x and y, there is a unique segment $[x, y]_U \in Hof(U)$. However it is not true in general that a set belong to $Hof^{\mathbb{Z}}(U)$ iff with any pair x, y of the set contains the segment $[x, y]_U$ as well (except the case $U = \mathbb{A}_n$). Because $Edm(U)$ is not stable under intersection, for pair of integer points x and y, we can not uniquely define a segment of $Edm(U)$. However, for pair of integer points x and y, we can define a set of segments as the set of minimal (wrt inclusion) polytopes (integer points of polytopes) of Edm(U) which contain x and y. The it i true that a set belongs $Edm(U)$ iff with any pair of its points it contains a segments for the pair. However, such a criterion has a drawback dealing with a problem of characterization of all U-segments for a given pair of points. (This is an open problem.)

Application to Economics with Indivisibles

Denote by $Edm^{\mathbb{Z}}(U)$ the class of sets of the form of integer points of polytopes of $Edm(U)$. Then we get

Theorem (Danilov-K-Murota (2001))

Let $\mathcal{E}$ be an economy with indivisibles and one divisible good (money). Suppose demand sets of agents belong to a $Edm^{\mathbb{Z}}(U)$ for some unimodular system U. Then there exists a competitive equilibrium.

One can get a dual existence theorem (in the spirit of Baldwin and Klemperer (2019))

Theorem

Let $\mathcal{E}$ be an economy with indivisibles and one divisible good (money). Suppose demand types of agents belong to a $Hof(U)$ for some unimodular system U. Then there exists a competitive equilibrium.

A collection $\mathcal{P}$ of polytopes is very ample if for any $P \in \mathcal{P}$, $nP \in \mathcal{P}$, $z + P \in \mathcal{P}$ for any $z \in \mathbb{Z}^n$, and any face of P belongs $\mathcal{P}$.
It occurs that classes $Edm(U)$ and $Hof(U)$ are the only classes of ample polytopes that satisfy the separation property. Namely we have the following theorem (Danilov and Koshevoy, 1998).

Theorem

Let $\mathcal{P}$ be an ample collection of integer polytopes. Then

- *$\mathcal{P}$ is stable under summation and the separation property holds if and only if there exists a totally unimodular system U such that $\mathcal{P} = Edm(U)$;*
- *$\mathcal{P}$ is stable under intersection and the separation property holds if and only if there exists a totally unimodular system U such that $\mathcal{P} = Hof(U)$;*

The following theorem characterizes unimodular systems U such that $Edm(U)$ is stable under intersections and $Hof(U)$ is stable under summations.

Theorem

Let U be a totally unimodular set such that $Edm(U)$ is stable under intersection. Then U is the direct sum of copies of $\mathbb{A}_1$ and $\mathbb{A}_2$.

For a given n, there are only finitely many totally unimodular systems (up to isomorphism) in $\mathbb{R}^n$ and maximal number of vectors in a totally unimodular systems is $\leq n(n+1)$.

Theorem (Danilov, Grishuhin, and K (2010))

Any full-dimensional totally unimodular system can be implemented as a collection of vectors of $2^{[n]}$ and their minuses.

For example, to implement the system $\mathbb{A}_n := \{A_n \cup \pm e_i\}$ as such a collection in $2^{[n]}$, we have to choose the following basis

$$e'_1 := e_1,\ e'_2 := e_1 + e_2, \ldots, e'_n = e_1 + e_2 + \ldots + e_n.$$

In such a basis we have the matrix realization of $\mathbb{A}_n$ is constituted of the columns of the form of intervals $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T$. (Note, that for the basis of $e_1, \ldots, e_k, -e_{k+1}, \ldots, -e_n$, we get the case considered by Sun and Yang.)

A characterization of totally unimodular system of vectors was obtained by Seymour (1980).

We need 'building blocks' and the following operations on totally unimodular systems.
In the matrix realization, the 0-sum of unimodular systems $U_1 = (I_k, M_1)$ and $U_2 = (I_l, M_2)$, k and l are dimensions of systems, and I_k denotes the diagonal matrix with 1's at the diagonal, $U_1 \oplus_0 U_2$ is represented by the matrix

$$\begin{pmatrix} I_k & 0 & M_1 & 0 \\ 0 & I_l & 0 & M_2 \end{pmatrix}$$

To construct 1-sum, we have to pick a common basis vector in unimodular systems U_1 and U_2, that is the <u>1-sum</u> $U_1 \oplus_1 U_2$ is represented by the matrix

$$\begin{pmatrix} I_{k-1} & 0 & 0 & M_1^{-k} & 0 \\ 0 & 1 & 0 & m_1^k & m_2^1 \\ 0 & 0 & I_{l-1} & 0 & M_2^{-1} \end{pmatrix}$$

To construct 2-sum, we consider unimodular systems U_1 and U_2 which have subsystems isomorphic to $\mathbb{A}_2$. In such a case, we take such subsystems a common subsystem in both U_1 and U_2, that is the <u>2-sum</u> $U_1 \oplus_2 U_2$ is represented by the matrix

$$\begin{pmatrix} I_{k-2} & 0 & 0 & 0 & 0 & M_1^{-\{k-1,k\}} & 0 \\ 0 & 1 & 0 & 0 & 1 & m_1^{k-1} & m_2^1 \\ 0 & 0 & 1 & 0 & 1 & m_1^k & m_2^2 \\ 0 & 0 & 0 & I_{l-2} & 0 & 0 & M_2^{-\{1,2\}} \end{pmatrix}$$

where $M_1 = \begin{pmatrix} 0 & M_1^{-\{k-1,k\}} \\ 1 & m_1^{k-1} \\ 1 & m_1^k \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & m_2^1 \\ 1 & m_2^2 \\ 0 & M_2^{-\{1,2\}} \end{pmatrix}$ (M_i has such a form because of our choice of the common subsystem isomorphic to $\mathbb{A}_2$).

Note, that 1-sum gives unimodular system in $\mathbb{R}^{k+l-1}$ and 2-sum in $\mathbb{R}^{k+l-2}$.

Graphic systems

To any (di)graph $G = (V, E)$ one can associate a unimodular system in $\mathbb{R}^{|V|}$, graphic unimodular system $\mathbb{A}(G)$, using the incidence matrix of G. Namely, let $A(G)$ be $|V| \times |E|$ incidence matrix, that is a column labeled by $e \in E$ has $+1$ at a row labeled by the emanating vertex of e, and -1 at a row labeled by terminal vertex of e, and 0 elsewhere. The the set of vectors corresponding to the columns of $A(G)$ and their minuses is the graphic unimodular system $\mathbb{A}(G)$.
A unimodular system U is maximal for any $r \notin U$ the system of vectors $U \cup r$ is not a unimodular.

The system A_n is maximal in $\mathbb{R}^{n-1}$.
Suppose that $r = (r_1, ..., r_n)$ is an integer vector such that $A_n \cup r$ is a unimodular system. We assert that for any $i \neq j$ there holds either $r_i r_j = 0$ or $r_i r_j = -1$. In fact, assume that $r_i r_j = 1$ holds for some $i \neq j$. Then consider the Abelian subgroup S generated by $e_i - e_j$, and r. The index of the subgroup S in $M := \mathbb{Z}^n \cap \{\sum x_i = 0\}$ is equal to the determinant of the matrix $\begin{pmatrix} r_i & 1 \\ r_j & -1 \end{pmatrix}$, that is ± 2. That contradicts to the unimodularity of S. Therefore, r has at most two non-zero coordinates, and in such a case these coordinates are of opposite signs. That is $r \in A_n$.

Cographic systems

To any graph G one can associate another unimodular system, the so called cographic unimodular system $\mathbb{D}(G)$. Namely choose a basis of the vectors of $\mathbb{A}(G)$. We get a totally unimodular matrix $(I_d, \hat{A})$ as decomposition of the vectors corresponding columns of $A(G)$ on this basis. The system of vectors corresponding to plus-minus the columns of $(I_{|E|-d}, \hat{A}^T)$ constitutes the cographic unimodular system $\mathbb{D}(G)$. Cubic (or 3-valent) graphs gives the most interesting examples of cographic systems.

Example. Consider a cubic graph, the bipartite graph $K_{3,3}$, then the incidence matrix is

$$A(K_{3,3}) := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

Let us choose the following columns as a basis: let e_1 be the first column, e_2 be the forth column, e_3, e_4, e_5 be the last three columns 7, 8, and 9, respectively. In this basis we have the following representation of $A(K_{3,3})$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Hence

$$\mathbb{D}(K_{3,3}) := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

Note, that $\mathbb{D}(K_{3,3})$ is an extension of the transportation matrix $T_{2,2}$ by one vector.

Let us represent the maximal system $\mathbb{A}_4$ isomorphic to A_5 (in dimension 4), where

$$\mathbb{A}_n := \{e_i - e_j \,|, i, j = 0, 1, \ldots, n\}, \text{ where } e_0 := 0,$$

as the following subsystem of vectors in the unit cube and their minuses

$$\mathbb{A}_4 := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

One can observe that the latter has one more vector. These systems are different because one can not add a $\{-1, 0, 1\}$-vector to the system $\mathbb{D}(K_{3,3})$ and still have a totally unimodular system. Hence $\mathbb{D}(K_{3,3})$ is not a graphic system.

The exceptional unimodular system R_{10} in dimension 5 which is neither graphic no cographic.

R_{10} consists of the following 21 vectors: 0, $\pm e_i$, $i = 1, \ldots, 5$, $\pm(e_1 - e_2 + e_3)$, $\pm(e_2 - e_3 + e_4)$, $\pm(e_3 - e_4 + e_5)$, $\pm(e_4 - e_5 + e_1)$, $\pm(e_5 - e_1 + e_2)\}$.

According to the Seymour theorem, every totally unimodular system can be constructed as 0-sums, 1-sums and 2-sums of graphic systems, cographic systems, and the system R_{10} .

Danilov and Grishuhin (1999) proposed a refinement of the Seymour theorem and characterized maximal totally unimodular systems. For example, 0-sums, 1-sums and 2-sums of graphic systems are always graphic and not maximal.
In dimension 2 and 3 all maximal totally unimodular systems are isomorphic to the graphic systems $\mathbb{A}_2$, $\mathbb{A}_3$, in dimension 4, there are two non-isomorphic maximal totally unimodular systems, namely $\mathbb{A}_4$ and $\mathbb{D}(K_{3,3})$. In dimension 5 there are 4, in dimension 6 there are 10.

For a totally unimodular system U, let $\mathcal{C}(U) := \{u^T x = 0,\ u \in U\}$ be a central hyperplane arrangement. Denote by $U^\perp$ the set of primitive vectors of one-dimensional cones (rays) of this arrangement.

Then a collection $\mathcal{L} \subset U^\perp$ is U-laminar if

- **$\mathcal{L}$ is totally unimodular system;**
- **$\mathcal{L}^\perp \subset U$.**

For any U-laminar collection $\mathcal{L}$, we have

$$Hof(\mathcal{L}) \subset Edm(U).$$

Discretely convex functions

Pseudo-convexity. Let f be a function on the lattice $\mathbb{Z}^n$ (or on some subset $D \subset \mathbb{Z}^n$). Similarly to the case with sets in $\mathbb{Z}^n$, the pseudo-convexity is the first approximation to discrete convex functions. Specifically, an affine function l is called a subdifferential of f at a point x if $l \leq f$ and $l(x) = f(x)$.

Proposition. For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, the following two properties are equivalent:
(1) f has a subdifferential at each point of its domain;
(2) f is the restriction to $\mathbb{Z}^n$ of a convex function defined on $\mathbb{R}^n$.

A function which satisfies the conditions of this proposition is pseudo-convex.

For a unimodular system $U \subset \mathbb{Z}^n$, we introduce the following two subclasses of (integer-valued) pseudo- convex functions:

$$DC^E_{\mathbb{Z}}(U) := \{f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \,|\, f(\mathbb{Z}^n) \subset \mathbb{Z} \cup \{+\infty\},$$
$$\operatorname{Arg}\max_{x \in \mathbb{R}^n}(p^T x - f(x)) \in Edm(U) \,\forall\, p \in \mathbb{R}^n\}.$$

$$DC^H_{\mathbb{Z}}(U) := \{g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \,|\, g(\mathbb{Z}^n) \subset \mathbb{Z} \cup \{+\infty\},$$
$$\operatorname{Arg}\max_{p \in \mathbb{R}^n}(x^T p - g(p)) \in Hof(U) \,\forall\, x \in \mathbb{R}^n\}.$$

We have to consider $DC^E_{\mathbb{Z}}(U)$ and $DC^H({}_{\mathbb{Z}}U)$ in dual to each other spaces. (For economics, $DC^E(U)$ are utility(-cost) functions, affinity areas of such functions take the form of demands, and affinity areas of $DC^H(U)$ are the prices supporting demand sets.)
Because $Edm(U)$ is stable wrt summation, $DC^E(U)$ is stable wrt <u>infimal convolution</u>

$$(f_1 * f_2)(x) = \inf_y (f_1(y) + f_2(x - y)),$$

and since $Hof(U)$ is stable wrt intersection, $DC^H(U)$ is stable wrt the summation.

Recall that the Legendre-Fenchel transformation of a convex function f is convex function

$$f^*(p) = \max_{x \in \mathbb{R}^n} (p^T x - f(x)).$$

The characterization Theorem suggested to think that classes $Edm(U)$ and $Hof(U)$ are dual. This is formalized as follows.

Theorem

For a totally unimodular system U, the Legendre-Fenchel transformation provides a bijection between $DC^E_{\mathbb{Z}}(U)$ and $DC^H_{\mathbb{Z}}(U)$. For $f_1, f_2 \in DC^E(U)$, we have

$$(f_1 * f_2)^* = f_1^* + f_2^*.$$

This generalizes duality results by Murota (2003).

An consequence of the duality, we have the separation

Theorem

Let $f_i : \mathbb{Z}^n \to \mathbb{R}(\mathbb{Z}) \cup \{+\infty\}$, $i = 1, 2$. Suppose $f_1 \geq -f_2$ and $f_i \in DC^E(U)$ ($DC^H(U)$). Then there exists separation affine function l, $f_1 \geq l \geq -f_2$. (l is integer if functions are integer-valued.)

And the following criterion of minimization of functions of $DC^E(U)$ and $DC^H(U)$ (compare with SI property by Gul and Stacchetti for matroids).

Theorem

a) Let $f \in DC^E(U)$ and $x \in dom_{\mathbb{Z}} f$. Then $f(x) \leq f(y)$, $y \in M$, if and only if, for any $u \in U$, there holds

$$f(x) \leq f(x + u). \tag{1}$$

b) Let $f \in DC^H(U)$-convex function and $p \in dom_{\mathbb{Z}} f$. Then $f(p) \leq f(q)$, $q \in M^*$, if and only if, for any $\xi \in U^{\perp}$, there holds

$$f(p) \leq f(p + \xi). \tag{2}$$

Note that complexity of checking (1) is polynomial since unimodular systems are polynomial, the complexity of checking (2) is also polynomial, but rather non-trivial relying on polynomiality of minimization submodular functions.
An analogy between discrete convexity of functions of $DC^H(U)$ and usual convexity is provided by

Theorem

Let $f \in DC^H(U)$. Then, for any $p, q \in (\mathbb{Z}^n)^*$ and any $p' \in [p, q]_U$, there holds

$$f(p) + f(q) \geq f(p') + f(q - (p' - p)). \tag{3}$$

Note, that for $U = A_n$, we can say if and only if in the above theorem.
In general, we have the following

Theorem

Let a function $f : (\mathbb{Z}^n)^* \to \mathbb{R}^{\{+\infty\}}$ and all its convolutions, $f \star f$, $f \star f \star f, \ldots$ satisfy (3). Then $f \in DC^H(U)$.

For $\mathbb{A}_n$, quadratic function are first interesting instance of DC functions. We have

- A function
$$f(x_1, \ldots, x_n) = \sum_{ij} a_{ij}\, x_i\, x_j$$
belongs $DC^H(\mathbb{A}_n)$ if and only if the following two conditions are met:
$$\text{a) } a_{ij} \leq 0 \;\text{ for } i \neq j, \text{and}$$
$$\text{b) } \sum_j a_{ij} \geq 0 \;\text{ for } i = 1, \ldots, n.$$
- A function
$$f(x_1, \ldots, x_n) = \sum_{ij} a_{ij}\, x_i\, x_j$$
belongs $DC^E(\mathbb{A}_n)$ if and only if
$$\text{a) } a_{ij} \leq 0 \;\text{ for } i \neq j, \text{and}$$
$$\text{b) } a_{ij} \geq \min(a_{ik}, a_{kj}) \;\text{ for all } i \neq j \neq k.$$

Thus, we have

Theorem. Suppose that buyers have utilities of the form $\sum_{ij} a^b_{ij}\, x_i\, x_j$, $b \in B$, such that, for each $b \in B$, it holds

$$\text{a) } a^b_{ij} \leq 0 \;\text{ for } i \neq j, \text{and}$$

$$\text{b) } a^b_{ij} \geq \min(a^b_{ik}, a^b_{kj}) \;\text{ for all } i \neq j \neq k.$$

Then the economy has competitive equilibria at any initial endowment.

Package functions

A package is a non-empty subset of the set I and we shall identify it with the bundle $1_A = [A] = \sum_{i \in A}[i]$.

Definition. Let $A \subset I$ be a package. An elementary A-package function is a function $u : \mathbb{Z}_+^I \to \mathbb{R}$ taking the following form

$$u(x) = v \ \min(1, x_i, \ i \in A) = \begin{cases} v, & \text{if } x \geq [A] \\ 0, & \text{otherwise,} \end{cases}$$

where v is a non-negative real number ('a reservation value').

A consumer endowed with an A-package utility views the items from A as strict complements. As a consequence the consumer derives a utility amount equal to v out of the consumption of the unitary bundle $[A]$ (or, for the matter, from any bundle with larger amounts of each item than in $[A]$). And consequently, this consumer does not derive any satisfaction from a bundle in which some item i from A would be missing.

The demand of this consumer is easy to figure out. The consumer demands package $[A]$ as soon as its cost $p(A) = \sum_{i \in A} p(i)$ is smaller than v, and demands no package $\{0\}$ when $p(A) > v$. In the boundary case $p(A) = v$, the consumer's demand is the set $D(u, p) = \{0, [A]\}$. Note that the consumer might be inclined to demand any amount of an item i, when the latter's price is 0.

We now move on to utility functions obtained as convolutions of elementary package functions. Let $\mathcal{T}$ be a collection of packages.

A function $f : \mathbb{Z}_+^I \to \mathbb{R}$ is a $\mathcal{T}$-package function (or is adapted to $\mathcal{T}$) if f is the convolution of a family of elementary A-package functions with $A \in \mathcal{T}$.

For example, convolution of two elementary A-package functions takes the form $(v_1 u^A) * (v_2 u^A) = \max(v_1, v_2) u^A$.
The function $f(x) = \varphi(\min(x_i,\ i \in I))$, where $\varphi : \mathbb{Z}_+ \to \mathbb{R}_+$ is a pseudo-concave function of one variable and $\varphi(0) = 0$, is compatible with the singleton family $\{I\}$. Indeed, it is the convolution of the following family $(v_n \min(1, x_i), n \in \mathbb{Z}_+)$ of I-package functions, where $v_n = \varphi(n+1) - \varphi(n)$. Conversely, an elementary I-package function has the form $\varphi(\min(x_i))$, where $\varphi(t) = v \min(1, t)$ for $t \geq 0$.
It is clear that if every buyer $b \in B$, in an economy, is equipped with a utility function u_b adapted to $\mathcal{T}$, then the aggregate utility function $U = *_b u_b$ is adapted to $\mathcal{T}$ as well.

Elementary package functions are pseudo-concave. However, taking any arbitrary collection of packages $\mathcal{T}$, and computing the associated $\mathcal{T}$-package function, we often enough end-up with a function that is not pseudo-concave. Hence, in general, a pure exchange economy with $\mathcal{T}$-package preferences will fail to exhibit equilibria. Here is a simple, but instructive example.

Example. Consider a pure exchange economy with three consumers a, b, c. Let I consist of three items, 1, 2 and 3. Now consider the following collection $\mathcal{T} := \{(1,2),(1,3),(2,3)\}$ of elementary packages. Assume that the consumers are endowed with the three elementary package utility functions: $u_a = 2\ \min(1, x_1, x_2)$, $u_b = 2\ \min(1, x_1, x_3)$ and $u_c = 2\ \min(1, x_2, x_3)$. Suppose that the initial endowment consists in a unique exemplar of each item, $[1] + [2] + [3]$. This economy has no competitive equilibria.

Here is the reason. By symmetry arguments, we may without loss of generality assume that $p(1) = p(2) = p(3) = p$. Let us now analyze the behavior of the aggregate demand in terms of p. If $p < 1$, then every buyer requests his/her elementary package; the aggregate demand consists in two units of each item and this is larger than the initial endowment. If $p > 1$, each individual's demand is equal to 0, and this will not yield an equilibrium either. Thus the only possible candidate to an equilibrium price is $p = 1$. At this price vector, each buyer is indifferent between buying his package or buying nothing. Computing the aggregate demand for all possible configurations, we easily notice that it never contains the initial endowment. Indeed, the demand of each buyer is limited to an even number of items: 2 or 0. Thus the aggregate demand will also consist of an even number of items and on the other hand the initial endowment encompasses an odd number of items. Thus there is no price for which the aggregate demand matches the aggregate endowment.

We now provide a criterion to assess the pseudo-concavity of a $\mathcal{T}$-package function. We associate to any family $\mathcal{T}$ the incidence matrix $M(\mathcal{T}) = (m_{i,A})$, rows correspond to elements of I, whereas columns correspond to sets from $\mathcal{T}$, defined as follows. For $i \in I$ and $A \in \mathcal{T}$, $m_{i,A}$ is equal to 1, if $i \in A$, and is equal to 0 otherwise. For instance, in the Example above M looks like this

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note, that the family of packages considered in the preceding Example is not unimodular, because $\det(M) = -2$.

Proposition

Let $\mathcal{T}$ be a collection of packages. The following two assertions are equivalent:
1) the collection $\mathcal{T}$ is unimodular;
2) $\mathcal{T}$-package functions are pseudo-concave.

Since each full-dimensional totally unimodular system can be implemented as a collection of vectors of $2^{[n]}$ and their minuses, we can implement any unimodular systems for using in the above proposition.
We get the following theorem.

Theorem

Let $\mathcal{T}$ be a unimodular collection of packages. Suppose that buyers have utilities, that are compatible with $\mathcal{T}$. Then the economy has competitive equilibria at any initial endowment.

Here are examples of networks which possess equilibria with agents which have utility functions from classes related to not graphic no cographic unimodular systems.

Example 1. There are 3 traders, the space of deals is 6-dimensional, which we illustrate pictorially

$$\begin{array}{ccc} \cdot & *_1 & *_2 \\ *_3 & \cdot & *_4 \\ *_5 & *_6 & \cdot \end{array}$$

The space of trades of first trader is $\mathbb{Z}^{*_1,*_2,*_3,*_5}$, the second operates of the space $\mathbb{Z}^{*_1,*_3,*_4,*_6}$, and the third one is of $\mathbb{Z}^{*_2,*_4,*_5,*_6}$.

Let the one-dimension demands of the first trader form graphic unimodular system isomorphic to $\mathbb{A}_4$ constituted from vectors $\{\pm e_1, \pm e_2, \pm e_3, \pm e_5, \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(e_2 + e_5), \pm(e_3 + e_5), \pm(e_1 + e_5)\}$.
Let the one-dimension demands of the third trader form cographic unimodular system also isomorphic to $\mathbb{W}_2 := \mathbb{D}(K_{3,3})$ constituted from vectors $\{\pm e_2, \pm e_4, \pm e_5, \pm e_6, \pm(e_2 + e_4), \pm(e_4 + e_6), \pm(e_2 + e_5), \pm(e_5 + e_6), \pm(e_2 + e_4 + e_6 + e_5)\}$.
These two cographic systems have a common subsystem $\{\pm e_2, \pm e_5, \pm(e_2 + e_5)\}$ isomorphic to $\mathbb{A}_2$. Thus, the union $A_4 \cup W_2$ is nothing but the 2-sum $A_4 \oplus_2 W_2$ and, hence, is a unimodular system in $\mathbb{R}^5$. This system is maximal and not isomorphic to graphical or cographical (Grishykhin V. and V. Danilov (1999), Section 9)
For one-dimension demands of the second trader we take the following subsystem of $A_4 \cup W_2$

$$\{\pm e_1, \pm e_3, \pm e_4, \pm e_6, \pm(e_1 - e_3), \pm(e_4 + e_6), \}$$

which is isomorphic to 0-sum of two $\mathbb{A}_2$'s.

Due to our theorem, in this example stable networks exist, and, from economic viewpoint, agents have utilities with mix of substitutes and complementarities, but since the underlying unimodular system is maximal and not isomorphic to graphical, this example can not be put in the frame of examples considered in the literature on matchings and networks.

This example can be easy extended to any number of agents in the network model.

Namely, as a corollary, we get that $n-1$ agents with full substitutes (utilities of agents are of $DC^E(\mathbb{A}_n)$) do not imply that the last one has to have substitute.Namely, we can get a system $\mathbb{A}_{n^2-n-9} \oplus_2 \mathbb{W}_2$ underlying for Trading network, which is not isomorphic to graphical or cographical.

Example 2. There are 3 traders, the space of deals is 6-dimensional, which we illustrate pictorially

$$\begin{array}{ccc} \cdot & *_1 & *_2 \\ *_3 & \cdot & *_4 \\ *_5 & *_6 & \cdot \end{array}$$

The space of trades of first trader is $\mathbb{Z}^{*_1,*_2,*_3,*_5}$, the second operates of the space $\mathbb{Z}^{*_1,*_3,*_4,*_6}$, and the third one is of $\mathbb{Z}^{*_2,*_4,*_5,*_6}$.
Let the one-dimension demands of the first trader form cographic unimodular system isomorphic to $\mathbb{D}(K_{3,3})$ constituted from vectors $\mathbb{W}_1 := \{\pm e_1, \pm e_2, \pm e_3, \pm e_5, \pm(e_1 + e_2), \pm(e_1 + e_3), \pm(e_2 + e_5), \pm(e_3 + e_5), \pm(e_1 + e_2 + e_3 + e_5)\}$.
Let the one-dimension demands of the third trader form cographic unimodular system also isomorphic to $\mathbb{D}(K_{3,3})$ constituted from vectors $\mathbb{W}_2 := \{\pm e_2, \pm e_4, \pm e_5, \pm e_6, \pm(e_2 + e_4), \pm(e_4 + e_6), \pm(e_2 + e_5), \pm(e_5 + e_6), \pm(e_2 + e_4 + e_6 + e_5)\}$.
These two cographic systems have a common subsystem $\{\pm e_2, \pm e_5, \pm(e_2 + e_5)\}$ isomorphic to $\mathbb{A}_2$. Thus, the union $W_1 \cup W_2$ is nothing but the 2-sum $W_1 \oplus_2 W_2$ and, hence, is a unimodular system.

This system is maximal and not isomorphic to graphical or cographical. For one-dimension demands of the second trader we take the following subsystem of $W_1 \cup W_2$

$$\{\pm e_1, \pm e_3, \pm e_4, \pm e_6, \pm(e_1 + e_3), \pm(e_4 + e_6), \}$$

which is isomorphic to 0-sum of two $\mathbb{A}_2$'s.
In this example stable networks exist, and, from economic viewpoint, agents have utilities with flavor of complementarities, but since the underlying unimodular system is maximal and not isomorphic to graphical, this example can not be put in the frame of examples yet considered.
This example can be easy extended to any number of agents in the network model.
Note, that the unimodular systems of these examples are non isomorphic.

Example 3. Consider matching with 5 buyers and 4 sellers. Buyers have demands with full substitutes on individual 4-dimensional spaces, $\mathbb{R}^{*1,*2,*3,*4}$ for the first buyer, $\mathbb{R}^{*5,*6,*7,*8}$ for the second buyer and so on. That is one-dimensional demands of the first buyer is the set isomorphic $\mathbb{A}_5$, and the same for others. We represent the space of trades pictorially as

$*_1$	$*_5 *_1$	$*_9$	$*_{13}$	$*_{17}$
$*_2$	$*_6$	$*_{10}$	$*_{14}$	$*_{18}$
$*_3$	$*_7$	$*_{11}$	$*_{15}$	$*_{19}$
$*_4$	$*_8$	$*_{12}$	$*_{16}$	$*_{20}$

Buyers have demands with full substitutes on individual 4-dimensional spaces labeled by columns, $\mathbb{R}^{*1,*2,*3,*4}$ for the first buyer, $\mathbb{R}^{*5,*6,*7,*8}$ for the second buyer and so on. That is one-dimensional demands of the first buyer is the set isomorphic $\mathbb{A}_4$, and the same for others.
Sellers have demands on 5-dimensional spaces labeled by rows $\mathbb{R}^{*1,*5,*9,*13,*17}$, $\mathbb{R}^{*2,*6,*10,*14,*18}$, and so on.

Let one-dimensional demands of each seller constitutes the exceptional unimodular system $\mathbb{R}^{10}$.
Let us consider 1-sums of these 4 systems isomorphic to $\mathbb{R}^{10}$ and 5 systems isomorphic to $\mathbb{A}_4$ along vectors e_i, $i = 1, \ldots, 20$. Then we get an unimodular system, and this system is not graphcial and cographical.
Thus, in such a model stable matchings exist.
This example shows that from the assumption on substitutable demands of buyers does not imply that seller have to have 'substitutable' supplies ('substitutable' means to form unimodular system isomorphic to $\mathbb{A}_n$ corresponding to substitutes).

Laminar functions

Let $\mathcal{L} \subset U^{\perp}$ be a laminar collection.
For each $I \in \mathcal{L}$ consider a convex function $f_I : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. Then we have

Theorem

The function

$$f_{\mathcal{L}}(x) = \sum_{I \in \mathcal{L}} f_I(I^T x)$$

belongs to $DC^E(U)$.

For a set of laminar collections $\mathcal{L}_1, \ldots, \mathcal{L}_k$, we have

$$f_{\mathcal{L}_1} * \cdots * f_{\mathcal{L}_k} \in DC^E(U).$$

From duality we have

$$f^{\mathcal{L}}(p) = *_{I \in \mathcal{L}}(f_I(I^T \cdot))^*$$

belongs to $DC^H(U)$, and $f^{\mathcal{L}_1} + \cdots + f^{\mathcal{L}_k} \in DC^H(U)$.

Maximal U-laminar collections for $U = \mathbb{A}_n$.

In such a case, $U^{\perp} = \pm 2^{[n]}$, and a collection $\mathcal{L} \subset 2^{[n]}$ is laminar, if, for any A, $B \in \mathcal{L}$, $A \cap B \neq \emptyset$ implies either $A \subseteq B$ or $B \subseteq A$. It is an easy fact that any laminar set is totally unimodular. Moreover, for any laminar $\mathcal{L}$, the following inclusion holds true

$$Hof(\mathcal{L}) \subset Edm(\mathbb{A}_n).$$

This endows us with a subclass of $Edm(\mathbb{A}_n)$, g-polymatroids, without checking the strong pair condition. Namely for a laminar $\mathcal{L}$, a polytope

$$\{a_A \leq 1_A^T x \leq b_A,\ a_A \leq b_A,\ A \in \mathcal{L}\}$$

is a g-polymatroid.
There are finitely many laminar systems in dimension n, and size of each bounded by $2n$. Recall, that Edmonds (1967) proved that union of two laminar systems is a totally unimodular system. For example, this implies that the matrix $T_{n,m}$ is totally unimodular.

Because of the Edmonds theorem, for $\mathcal{T} = \mathcal{L}_1 \cup \mathcal{L}_2$, in economies with indivisibles with buyers which have utilities compatible with such $\mathcal{T}$ there exists an equilibrium.
Note that in the above non-existence example we have the case with three laminar collections.

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